Tampere University
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Resource-efficient NextG MIMO Sensing and Comms

Bridging Sparse Arrays, Spatial Modulation and Subspace Codes

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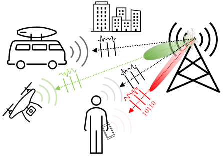
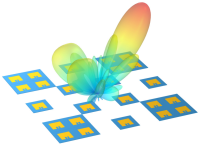
Pronounced role of array geometry in NextG

- Sensing increasingly relevant in next-generation (NextG) wireless—chief driver: ISAC
- Spatio-temporal resources should be used efficiently, serving both sensing and comms

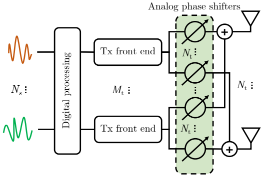




Time ↓

$$\begin{bmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,N} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{T,1} & S_{T,2} & \cdots & S_{T,N} \end{bmatrix}$$

Space ⟶



Harnessing physical-layer (design and signal processing) key in NextG MIMO

RF convergence and power/cost/hardware limitations call for rethinking design and use of array geometry, transmit waveforms, modulation, coding. . .

Andrews, J. G., Humphreys, T. E., and Ji, T. (2024). 6g takes shape. *IEEE BITS the Information Theory Magazine*, 4(1):2–24

Liu, F., Cui, Y., Masouros, C., Xu, J., Han, T. X., Eldar, Y. C., and Buzzi, S. (2022). Integrated sensing and communications: Toward dual-functional wireless networks for 6G and beyond. *IEEE Journal on Selected Areas in Communications*, 40(6):1728–1767

Björnson, E., Chae, C.-B., Jr., R. W. H., Marzetta, T. L., Mezghani, A., Sanguinetti, L., Rusek, F., Castellanos, M. R., Jun, D., and Özlem Tugfe Demir (2024). Towards 6G MIMO: Massive spatial multiplexing, dense arrays, and interplay between electromagnetics and processing

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Rajamäki, R. and Pal, P. (2024). Sparse array sensor selection in ISAC with identifiability guarantees. In *58th Asilomar Conference on Signals, Systems and Computers*, pages 1–5, Asilomar, Pacific Grove, CA, USA

https://arxiv.org/abs/2412.21002

- **Part II: Subspace Coding Meets Sensing**

Mahdavifar, H., Rajamäki, R., and Pal, P. (2024). Subspace coding for spatial sensing. In *IEEE International Symposium on Information Theory (ISIT)*, pages 2394–2399

https://arxiv.org/pdf/2407.02963

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Part I: Index Modulation for ISAC with Sensing Guarantees

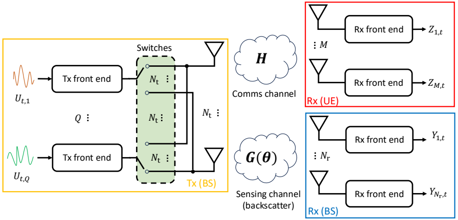
Integrated Sensing and Communications (ISAC) signal model

- Consider monostatic MIMO-ISAC system
- Comms: *Downlink* signal @ UE:

$$\boldsymbol{Z} = \boldsymbol{H}\boldsymbol{S}^\top + \boldsymbol{W}$$

- Sensing: *Backscattered* signal @ BS

$$\boldsymbol{Y} = \boldsymbol{G}(\boldsymbol{\theta})\boldsymbol{S}^\top + \boldsymbol{N}$$



- Goals: *Decode* $\boldsymbol{S}$ given $\boldsymbol{H}$ (comms) and *estimate* target directions $\boldsymbol{\theta}$ given $\boldsymbol{S}$ (sensing)

Same transmit waveform $\boldsymbol{S}$ and array geometry $\mathbb{D}_t$ used for both sensing and comms.
In particular, any *subset of Tx sensors selected* are also shared by both tasks.

Sensor selection in MIMO communications and ISAC

- Sensor selection widely considered in both ISAC and classical MIMO communications
- Typical use case: Information encoding in Tx subarrays, a.k.a. "spatial/index modulation"

- Di Renzo, M., Haas, H., Ghrayeb, A., Sugiura, S., and Hanzo, L. (2014). Spatial modulation for generalized MIMO: Challenges, opportunities, and implementation. *Proceedings of the IEEE*, 102(1):56–103
- Datta, T., Eshwaraiah, H. S., and Chockalingam, A. (2016). Generalized space-and-frequency index modulation. *IEEE Transactions on Vehicular Technology*, 65(7):4911–4924
- Hassanien, A., Amin, M. G., Zhang, Y. D., and Ahmad, F. (2016). Signaling strategies for dual-function radar communications: an overview. *IEEE Aerospace and Electronic Systems Magazine*, 31(10):36–45
- Basar, E., Wen, M., Mesleh, R., Di Renzo, M., Xiao, Y., and Haas, H. (2017). Index modulation techniques for next-generation wireless networks. *IEEE Access*, 5:16693–16746
- Wang, X., Hassanien, A., and Amin, M. G. (2019). Dual-function MIMO radar communications system design via sparse array optimization. *IEEE Transactions on Aerospace and Electronic Systems*, 55(3):1213–1226
- Shamasundar, B. and Nosratinia, A. (2022). On the capacity of index modulation. *IEEE Transactions on Wireless Communications*, 21(11):9114–9126
- Ma, D., Huang, T., Shlezinger, N., Liu, Y., and Eldar, Y. C. (2023). *Index Modulation Based ISAC*, pages 241–268. Springer Nature Singapore, Singapore
- Xu, J., Wang, X., Aboutanios, E., and Cui, G. (2023). Hybrid index modulation for dual-functional radar communications systems. *IEEE Transactions on Vehicular Technology*, 72(3):3186–3200
- Shamasundar, B. and Nosratinia, A. (2024). Index modulation with channel training: Spectral efficiency and optimal antenna alphabets. *IEEE Transactions on Wireless Communications*, 23(3):2241–2252
- Elbir, A. M., Celik, A., Eltawil, A. M., and Amin, M. G. (2024). Index modulation for integrated sensing and communications: A signal processing perspective. *IEEE Signal Processing Magazine*, 41(5):44–55

Our focus: Which array geometries and Tx subarrays are suitable for ISAC (sensing/comms)?
Gap in analytical understanding of ISAC trade-offs when employing index modulation

MIMO communications via Tx sensor selection

- (Generalized) spatial modulation: Information encoded in Tx sensor subsets = codewords

(a) Tx array **(b)** “10” **(c)** “01” **(d)** “00”

- Advantages: low power consumption and cost, improved achievable rate...

Datta, T., Eshwaraiah, H. S., and Chockalingam, A. (2016). Generalized space-and-frequency index modulation. *IEEE Transactions on Vehicular Technology*, 65(7):4911–4924

Shamasundar, B. and Nosratinia, A. (2022). On the capacity of index modulation. *IEEE Transactions on Wireless Communications*, 21(11):9114–9126

Encoding information in Tx subarray selections

General form of waveform matrix

$$\boldsymbol{S} = \begin{bmatrix} s_{1,1} & s_{1,2} & \dots & s_{1,N_\mathrm{t}} \\ s_{2,1} & s_{2,2} & \dots & s_{2,N_\mathrm{t}} \\ \vdots & \vdots & \ddots & \vdots \\ s_{T,1} & s_{T,2} & \dots & s_{T,N_\mathrm{t}} \end{bmatrix}$$

Time ↓

Space ⟶

Waveform matrix performing sensor selection

$$[\boldsymbol{S}]_{\ell,:} = \underbrace{\begin{bmatrix} x_{\ell,1} & \dots & x_{\ell,Q} \end{bmatrix}}_{=\boldsymbol{x}_\ell^\top,\ \text{data symbols at time } \ell} \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & & & & 0 \\ \vdots & & & & & & & \\ 0 & & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}}_{=\boldsymbol{J}_{\mathbb{S}_\ell}^\top \in \{0,1\}^{Q\times N_\mathrm{t}},\ \text{rows correspond to subarray } \mathbb{S}_\ell \subseteq \mathbb{D}_\mathrm{t}}$$

- Subset of Q Tx sensors $\mathbb{S}_\ell \subseteq \mathbb{D}_\mathrm{t}$ activated to transmit data streams $\boldsymbol{x}_\ell$ at time ℓ
- Each Tx subarray $\mathbb{S}_\ell$ corresponds to a codeword in sensor selection codebook $\mathcal{C}$
- Comms Rx (UE) tries to decode subset $\mathbb{S} \in \mathcal{C}$ (and $\boldsymbol{x} \in \mathcal{X}$) given $(\boldsymbol{z}, \boldsymbol{H})$

$$(\hat{\mathbb{S}}, \hat{\boldsymbol{x}}) = \underset{\mathbb{S}' \in \mathcal{C}, \boldsymbol{x}' \in \mathcal{X}}{\arg\min} \|\boldsymbol{z} - \boldsymbol{H}\boldsymbol{J}_{\mathbb{S}'}\boldsymbol{x}'\|_2$$

From conventional MIMO communications codebook towards ISAC codebook

- For simplicity, we focus on ("slow-time") sensor selection: $\boldsymbol{S}^\top = \boldsymbol{J}_{\mathbb{S}}\boldsymbol{X}$, $\boldsymbol{X}$ known to Rx[1]
- Large codebook (many subarrays) desirable for high-rate communications (at high SNR)

$$\binom{N_\mathrm{t}}{Q} \text{ possible } Q \text{ antenna subsets (codewords), given } N_\mathrm{t} \text{ total Tx antennas}$$

- All codewords equivalent for communications if $\boldsymbol{H}$ unstructured (e.g. Rayleigh fading)

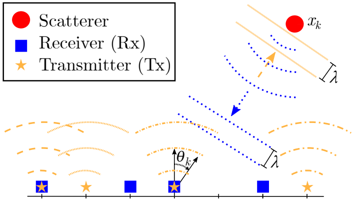
Which Tx sensor subsets $\mathbb{S} \subset \mathbb{D}_\mathrm{t}$ are suited for sensing and *how many* such subsets are there?
Answer crucially depends on structure of sensing channel $\boldsymbol{G}(\boldsymbol{\theta})$

[1] Wang, X., Hassanien, A., and Amin, M. G. (2019). Dual-function MIMO radar communications system design via sparse array optimization. *IEEE Transactions on Aerospace and Electronic Systems*, 55(3):1213–1226

Latent additive structure in signal model = "virtual array"

- Canonical sensing channel model highly structured:

$$\boldsymbol{G}(\boldsymbol{\theta}) = \boldsymbol{A}_{\mathbb{D}_\mathrm{r}}(\boldsymbol{\theta})\,\mathrm{diag}(\boldsymbol{\gamma})\boldsymbol{A}_{\mathbb{D}_\mathrm{t}}^\top(\boldsymbol{\theta}),$$

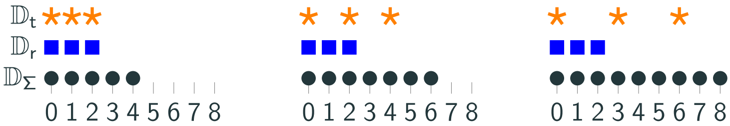


- Hidden additive structure of Tx-Rx sensor pairs:

$$[\boldsymbol{G}(\boldsymbol{\theta})]_{n,m} = \sum_{k=1}^{K} \gamma_k e^{j\pi(d_\mathrm{t}[m]+d_\mathrm{r}[n])\sin\theta_k}$$

- Measurements $\boldsymbol{Y}$ behave as if received by a virtual array ("co-array") defined by *sum set*:

$$\mathbb{D}_\Sigma \triangleq \mathbb{D}_\mathrm{t} + \mathbb{D}_\mathrm{r} = \{d_\mathrm{t} + d_\mathrm{r} \mid d_\mathrm{t} \in \mathbb{D}_\mathrm{t};\, d_\mathrm{r} \in \mathbb{D}_\mathrm{r}\}$$



Array redundancy: Elements of virtual array may overlap (unlike physical sensors)

Identifiability as sensing criterion for ISAC via Tx sensor selection

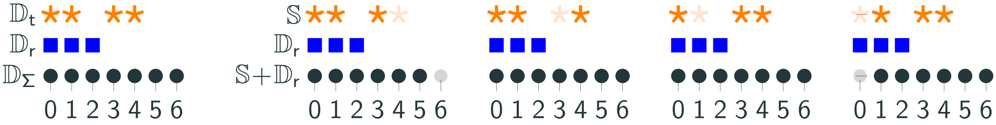
- To *uniquely* recover $\boldsymbol{\theta}$ given $\boldsymbol{Y}(\boldsymbol{\theta})$, *necessary* that $\boldsymbol{\theta} \mapsto \boldsymbol{Y}(\boldsymbol{\theta})$ is injective ($\boldsymbol{\theta}$ is identifiable)
- Up to $\frac{N_\Sigma}{2}$ targets identifiable if $\mathbb{D}_\Sigma$ is contiguous[2], i.e., $\mathbb{D}_\Sigma = \mathbb{U}_{N_\Sigma} \triangleq \{0, 1, \ldots, N_\Sigma - 1\}$

Goal: Characterize which of the $\binom{N_t}{Q}$ possible Tx subarrays $\mathbb{S} \subset \mathbb{D}_t$ *maximize identifiability* by having uniform (contiguous) sum set $\mathbb{S} + \mathbb{D}_r = \mathbb{U}_{N_\Sigma} = \{0, 1, \ldots, N_\Sigma - 1\}$

[2] Yang, Z., Li, J., Stoica, P., and Xie, L. (2018). Chapter 11 - sparse methods for direction-of-arrival estimation. In Chellappa, R. and Theodoridis, S., editors, *Academic Press Library in Signal Processing, Volume 7*, pages 509–581. Academic Press

Example: Sum set constraint prunes codebook

- Given $\mathbb{D}_t$, $\mathbb{D}_r$, s.t. $\mathbb{D}_\Sigma = \mathbb{U}_{N_\Sigma}$, which Q-sensor subarrays $\mathbb{S} \subset \mathbb{D}_t$ yield contiguous sum set?
- Example with $(Q, N_t, N_r, N_\Sigma) = (3, 4, 3, 7)$:



(a) $\mathbb{D}_t = \{0, 1, 2, 4\}$ **(b)** $\mathbb{S} = \{0, 1, 3\}$ ✘ **(c)** $\mathbb{S} = \{0, 1, 4\}$ ✓ **(d)** $\mathbb{S} = \{0, 3, 4\}$ ✓ **(e)** $\mathbb{S} = \{1, 3, 4\}$ ✘

In this example, only 2 out of 4 Tx subarrays (with 3 sensors) yield maximal identifiability.
Not all subarrays (or array geometries) are created equal from a sensing perspective!

Problem formulation: ISAC codebook with identifiability guarantees

- We constrain sum set of codewords to equal that of full array (sum co-array)

$$\mathcal{C}^{\mathrm{s}}(Q,\mathbb{D}_{\mathrm{t}},\mathbb{D}_{\mathrm{r}}) \triangleq \{\mathbb{S}\subseteq\mathbb{D}_{\mathrm{t}} : |\mathbb{S}| = Q,\ \mathbb{S}+\mathbb{D}_{\mathrm{r}} = \mathbb{D}_{\mathrm{t}}+\mathbb{D}_{\mathrm{r}}\}.$$

- Identifiability-maximizing codebook: given $(Q, N_{\mathrm{t}}, N_{\mathrm{r}}, N_\Sigma)$, optimized over $(\mathbb{D}_{\mathrm{t}}, \mathbb{D}_{\mathrm{r}})$

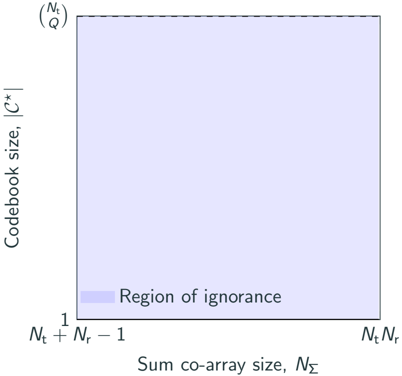
$$(\mathcal{C}^\star,\mathbb{D}_{\mathrm{t}}^\star,\mathbb{D}_{\mathrm{r}}^\star) \triangleq \operatorname*{arg\,max}_{\mathcal{C},\mathbb{D}_{\mathrm{t}},\mathbb{D}_{\mathrm{r}}}\{|\mathcal{C}| : \mathcal{C} = \mathcal{C}^{\mathrm{s}}(Q,\mathbb{D}_{\mathrm{t}},\mathbb{D}_{\mathrm{r}}), |\mathbb{D}_{\mathrm{t}}| = N_{\mathrm{t}}, |\mathbb{D}_{\mathrm{r}}| = N_{\mathrm{r}}, \mathbb{D}_{\mathrm{t}}+\mathbb{D}_{\mathrm{r}} = \mathbb{U}_{N_\Sigma}\}$$

- Each codeword in $\mathcal{C}^\star$ encodes $\lfloor \log_2 |\mathcal{C}^\star| \rfloor$ bits and guarantees identifiability of $\frac{N_\Sigma}{2}$ targets

What is the size of $\mathcal{C}^\star$ as a function of N_Σ and Q?
Challenging combinatorial optimization problem; solution(s) unknown. Bounds on $|\mathcal{C}^\star|$?

Note: receiver array geometry $\mathbb{D}_{\mathrm{r}}$ is fixed for all codewords (transmit subarrays)

Characterizing ISAC trade-off: $|\mathcal{C}^\star|$ as a function of (N_Σ, Q), given (N_t, N_r)



Can we narrow down range of possible values of $|\mathcal{C}^\star|$ from above naive bounds?

When is $\mathcal{C}^\star$ nonempty?

- $\forall N_\mathrm{t}, N_\mathrm{r} \in \mathbb{N}_+$, # of sum co-array elements N_Σ satsifies (max./min. redundant array)

$$N_\mathrm{t} + N_\mathrm{r} - 1 \leqslant N_\Sigma \leqslant N_\mathrm{t} N_\mathrm{r}$$

- Constraint $\mathbb{S} + \mathbb{D}_\mathrm{r} = \{0, 1, \dots, N_\Sigma - 1\}$ implies that $|\mathbb{S}||\mathbb{D}_\mathrm{r}| \geqslant |\mathbb{S} + \mathbb{D}_\mathrm{r}| = N_\Sigma$, i.e.,

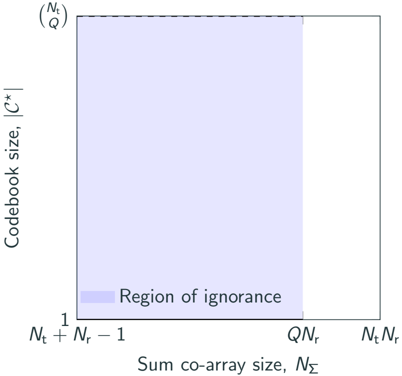
$$Q \geqslant L \triangleq \lceil N_\Sigma / N_\mathrm{r} \rceil$$

- This is a special case of a more general optimal operating point[3], $\mathrm{rank}(\boldsymbol{S}) = L$, where *low-rank* waveforms may fully leverage sum co-array of *redundant* arrays ($N_\Sigma < N_\mathrm{t} N_\mathrm{r}$)

$\mathcal{C}^\star(Q, N_\mathrm{t}, N_\mathrm{r}, N_\Sigma)$ is nonempty only if $N_\Sigma \in [N_\mathrm{t} + N_\mathrm{r} - 1, N_\mathrm{t} N_\mathrm{r}]$ and $Q \in [L, N_\mathrm{t}]$

[3] Rajamäki, R. and Pal, P. (2023). Importance of array redundancy pattern in active sensing. In *Ninth IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, pages 156–160

Narrowing down possible values of $|\mathcal{C}^\star|$



Can we find a tighter upper bound on $|\mathcal{C}^\star|$?

Upper bound on $|\mathcal{C}^\star|$: At least two sensors are shared by all admissible $\mathbb{S}$

- Observation: $\mathbb{S} \subseteq \mathbb{D}_\mathrm{t}$ must include *first* and *last* sensor of $\mathbb{D}_\mathrm{t}$ for $\mathbb{S} + \mathbb{D}_\mathrm{r} = \mathbb{D}_\mathrm{t} + \mathbb{D}_\mathrm{r}$, i.e.,

$$\mathbb{S} + \mathbb{D}_\mathrm{r} = \mathbb{D}_\mathrm{t} + \mathbb{D}_\mathrm{r} \text{ \textit{only} if } \mathbb{S} \supseteq \{\min \mathbb{D}_\mathrm{t}, \max \mathbb{D}_\mathrm{t}\}$$

Lemma (Upper bound on size of JCS codebook with identifiability guarantees)

For any admissible tuple $(Q, N_\mathrm{t}, N_\mathrm{r}, N_\Sigma)$, *we have* $|\mathcal{C}^\star(Q, N_\mathrm{t}, N_\mathrm{r}, N_\Sigma)| \leqslant \binom{N_\mathrm{t}-2}{Q-2}$.

As we will see, this simple bound is tight for the extreme points of N_Σ:
$N_\Sigma = N_\mathrm{t} + N_\mathrm{r} - 1$ (ULA Tx & Rx arrays) and $N_\Sigma = N_\mathrm{t} N_\mathrm{r}$ (nonredundant array)

(a) Uniform linear array (ULA) **(b)** Nonredundant (nested) array

Tightness of upper bound in case of ULA and nonredundant arrays

Corollary (ULA and nonredundant array)

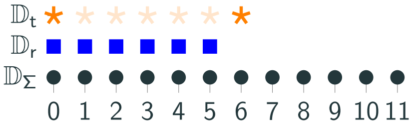
If $N_\Sigma = N_t + N_r - 1$ (ULA Tx and Rx) and $N_t \leqslant N_r + 1$, then for any $Q \in [2, N_t]$,

$$|\mathcal{C}^\star(Q, N_t, N_r, N_\Sigma)| = \binom{N_t - 2}{Q - 2}.$$

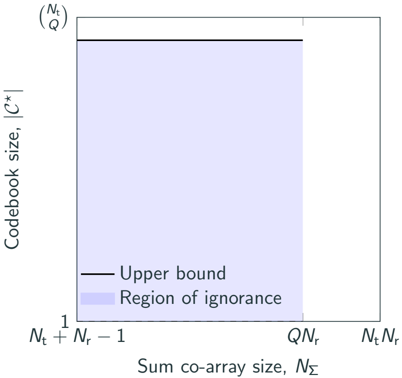
If $N_\Sigma = N_r N_t$ (nonredundant array), then $Q = N_t$ (subset selection impossible), which implies

$$|\mathcal{C}^\star(Q, N_t, N_r, N_\Sigma)| = 1.$$

Completely characterizes $|\mathcal{C}^\star|$ for ULA and nonredundant arrays: ULA has many codewords, nonredundant array just one (no comms)—ISAC codebook size-identifiability trade-off!



Narrowing down possible values of $|\mathcal{C}^\star|$



Can we find a tighter lower bound on $|\mathcal{C}^\star|$?

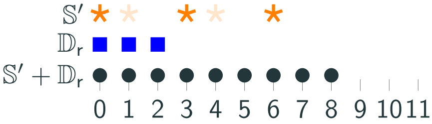
Tighter upper bounds for remaining values $N_\Sigma \in [N_t + N_r, N_t N_r - 1]$ topic of future work

Lower bound on $|\mathcal{C}^\star|$: Fix L sensors of all subarrays $\mathbb{S}$ to guarantee identifiability

Theorem (Lower bound on size of JCS codebook with identifiability guarantees)

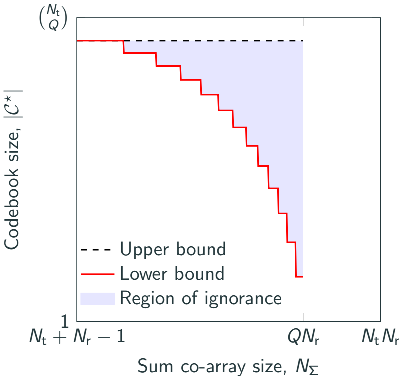
Suppose $(Q, N_\mathrm{t}, N_\mathrm{r}, N_\Sigma)$ *is admissible and* $L = \frac{N_\Sigma}{N_\mathrm{r}} \in \mathbb{N}_+$. *Then* $|\mathcal{C}^\star(Q, N_\mathrm{t}, N_\mathrm{r}, N_\Sigma)| \geqslant \binom{N_\mathrm{t}-L}{Q-L}$.

- Proof by construction: Set $\mathbb{D}_\mathrm{r} = \mathbb{U}_{N_\mathrm{r}}$ and $\mathbb{D}_\mathrm{t} \supseteq \mathbb{S} \supseteq \mathbb{S}' \triangleq \{0, N_\mathrm{r}, \ldots, (L-1)N_\mathrm{r}\}$



Lower bound based on *nested* sparse (sub)array design and is hence achievable

Narrowing down possible values of $|\mathcal{C}^\star|$

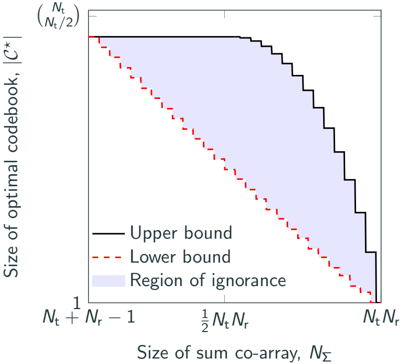


$$\binom{N_t - L}{Q - L} \leqslant |\mathcal{C}^\star(Q, N_t, N_r, N_\Sigma)| \leqslant \binom{N_t - 2}{Q - 2}$$

Upper and lower bounds can still be tightened for $N_t + N_r - 1 < N_\Sigma < N_t N_r$: future work!

Size of optimal sensor-selection-based ISAC codebook

- Optimal # of selected sensors Q? Choosing Q to maximize $|\mathcal{C}^\star|$ yields:



ISAC trade-off: Increased identifiability comes at expense of reduced codebook size

Contributions: sensor selection with identifiability guarantees for ISAC

We studied sensor selection for ISAC; specifically codebook design (codewords=Tx subarrays)

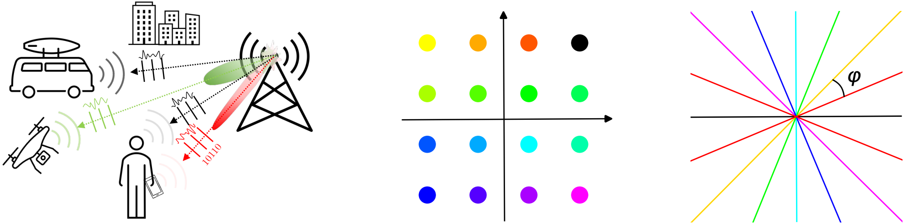
Main contributions:

1. First results on (sens. sel.) ISAC codebook design with identifiability guarantees
2. Established ISAC trade-off between codebook size (comms) and identifiability (sensing)
3. Showed that codebook of canonical uniform array contains many admissible codewords!

https://arxiv.org/abs/2412.21002

Part II: Subspace Coding Meets Sensing

Sensing meets coding in emerging applications: The subspace connection



(a) Sensing/ISAC scenario **(b)** 16-QAM **(c)** 1D subspace code

- Sensing of increasing importance in emerging wireless systems: E.g., channel estimation in mmWave, convergence of communications and sensing in 6G & beyond systems…
- Subspace codes encode information in *subspaces* (rather than, e.g., vectors); key application: noncoherent communications where channel is unknown at receiver

Hint of deep underlying connection between sensing and subspace coding

Sensing typically involves estimating parameters, such as directions-of-arrival (DoAs) of emitters / targets, *encoded in a subspace* of the received signal—akin to subspace coding!

Subspace coding and spatial sensing actively studied...albeit in isolation

Subspace coding:

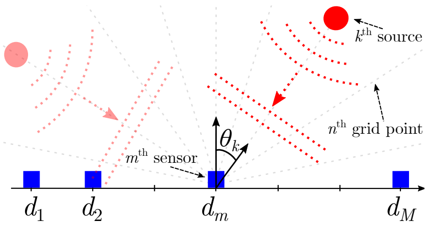
- Packings in Grassmannian space
  Shannon (1959); Conway et al. (1996); Calderbank et al. (1999); Barg and Nogin (2002); Fickus et al. (2018)
- Noncoherent MIMO communications
  Marzetta and Hochwald (1999); Agrawal et al. (2001); Zheng and Tse (2002); Strohmer and Heath Jr (2003)
- Limited feedback beamforming
  Mukkavilli et al. (2003); Love et al. (2003)
- Random network coding (in $\mathbb{F}_q^N$)
  Koetter and Kschischang (2008)
- Analog subspace codes (in $\mathbb{R}^N$ or $\mathbb{C}^N$)
  Soleymani and Mahdavifar (2022)

Spatial sensing:

- High-resolution (subspace) methods
  Schmidt (1986); Roy and Kailath (1989)
- Sparse sensor arrays
  Moffet (1968); Hoctor and Kassam (1990); Koochakzadeh and Pal (2016); Liu and Vaidyanathan (2017); Wang and Nehorai (2017); Amin (2024)
- Sensing-enabled emerging applications: mmWave, ISAC, autonomous sensing,...
  Alkhateeb et al. (2014); Sun, Petropulu, and Poor (2020); Ahmadipour, Kobayashi, Wigger, and Caire (2022); Chang, Wang, Erdoğan, and Bloch (2023)

Are these two *seemingly* different fields fundamentally linked? How?

Sensing model in direction-of-arrival (DoA) estimation



- Canonical sensing scenario: superposition of plane waves impinge on sensor array
- Spatial sensing receiver system model (M spatial samples, L temporal samples, K sources):

$$y_{m,l} = \sum_{k=1}^{K} e^{j\pi d_m \sin\theta_k} x_{k,l} + w_{m,l},\ m \in [M];\ l \in [L] \quad (1)$$

- Goal: Recover unknown DoAs $\{\theta_k\}_{k=1}^{K}$ given $y_{m,l}$, without knowledge of $x_{k,l}$ ($\forall m, k, l$)

We now show that spatial sensing can be interpreted as a subspace coding problem

"Message" in spatial sensing = unknown DoAs encoded in subspace

- Let $\boldsymbol{u} \stackrel{\text{def}}{=} (\theta_1, \theta_2, \ldots, \theta_K)$ denote "message" (DoAs) and rewrite system model as:

$$Y_{M\times L} = H(\boldsymbol{u})_{M\times K} X_{K\times L} + W_{M\times L} \tag{2}$$

- $\boldsymbol{u} \in \mathscr{M}$ and (source signal) X produced by nature—both fully unknown

Range space of $H(\boldsymbol{u})$ (denoted $\langle H(\boldsymbol{u})\rangle$) encodes information of interest $\boldsymbol{u}$

Spatial sensing typically only interested in subspace $\langle H(\boldsymbol{u})\rangle \stackrel{\text{rank}(X)=K}{=} \langle H(\boldsymbol{u})X\rangle \stackrel{W=0}{=} \langle Y\rangle$, *not* X

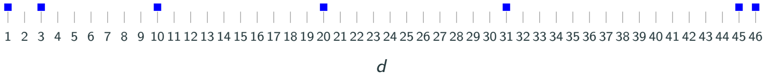
$|\mathscr{M}| = \binom{N}{K}$, assuming θ's are non-overlapping and belong to grid of size N

“Encoding function” in spatial sensing

- Encoding function $H(\cdot) : \mathscr{M} \mapsto \mathbb{C}^{M \times K}$ has harmonic structure due to physics

$$H_{m,k}(\boldsymbol{u}) = e^{j\pi d_m \sin\theta_k}, \text{ for } m \in [M]; k \in [K] \tag{3}$$

- Structure imposed by nature—not by design, as sometimes in comms (Hochwald et al., 2000)
- $H(\boldsymbol{u})$ can be influenced only via choice of sensing geometry $\{d_m\}_{m=1}^{M}$
  - Realized in hardware, e.g., via antenna placement (static) or selection (dynamic)



https://verontrustgroningen.nl/militair-wapen/, 08/02/25

Connection to subspace coding

- A sensing subspace code is then defined, given $\mathscr{M}$, $\{d_m\}_{m=1}^M$, and $H(\cdot)$ following (3), as

$$\mathcal{C} \overset{\text{def}}{=} \{\langle H(\boldsymbol{u})\rangle : \boldsymbol{u} \in \mathscr{M}\} \subseteq \mathbb{C}^M. \tag{4}$$

Key insight

$\mathcal{C}$ is a *structured* subspace code in $\mathbb{C}^M$; sensing geometry design a subspace coding problem!

What are *"good"* sensing subspace codes? How to construct such codes?

Next we partially answer these questions for $K = 1$. General case $K \geqslant 2$ ongoing work.

Sensing Subspace Codes: New Insights and (Near) Optimal Designs for Single Emitter

Metric for sensing subspace codes

- Analog subspace codes employ *subspace distance*[4] metric (Soleymani and Mahdavifar, 2022)

$$d^{(s)}(U, V) \overset{\text{def}}{=} \sum_{i=1}^{K} \sin^2(\beta_i), \tag{5}$$

  where β_i is the ith principal angle between K-dimensional subspaces U and V

- Minimum distance of analog subspace code $\mathcal{C}$ (suitable also for sensing subspace codes):

$$d^{(s)}_{\min}(\mathcal{C}) \overset{\text{def}}{=} \min_{U, V \in \mathcal{C}, U \neq V} d^{(s)}(U, V).$$

A large minimum distance $d^{(s)}_{\min}(\mathcal{C})$ facilitates a low probability of decoding error

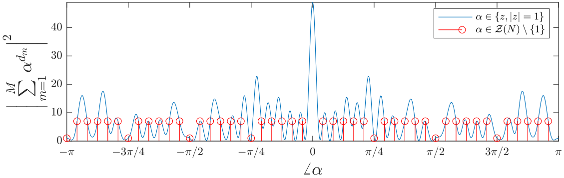
[4] A.k.a. squared chordal distance. Specifically, $\beta_i = \arccos |V_i^{\mathrm{H}} U_i|$, where $U_i \in U$, $V_i \in V$ are orthonormal basis vectors indexed such that $|V_i^{\mathrm{H}} U_i| \geqslant |V_i^{\mathrm{H}} U_\ell|$, $\forall \ell \neq i$.

Maximization of minimum distance when $K = 1$ ("line packing")

- Focus: single emitter case ($K = 1$) for insight into desirable sensing subspace codes; also important special case in both sensing and comms applications (Chiu et al., 2019; Liu et al., 2023)
- Sensing subspace code design problem becomes

$$\underset{\{d_m\}_{m=1}^M}{\text{maximize}}\ d_{\min}^{(s)}(\mathcal{C}) \overset{K=1}{=} \underset{\{d_m\}_{m=1}^M}{\text{minimize}}\ \max_{\substack{\alpha\in\mathcal{Z}(N)\\ \alpha\neq 1}} \left|\sum_{m=1}^{M} \alpha^{d_m}\right|^2,$$

where by choice of grid, $\alpha = e^{j\pi \sin\theta} \in \mathcal{Z}(N) \overset{\text{def}}{=} \{e^{j2\pi n/N}, n\in[N]-1\}$ are N-th roots of unity.



Interpretation: minimize maximum sidelobe of *sampled* beampattern (w.r.t. array geometry)

Difference sets and minimum distance

- Objective function term can be written as

$$\left|\sum_{m=1}^{M}\alpha^{d_m}\right|^2=\left(\sum_{i=1}^{M}\alpha^{d_i}\right)\left(\sum_{\ell=1}^{M}\alpha^{d_\ell}\right)^*=M+\sum_{i,\ell\in[M],i\neq\ell}\alpha^{d_i-d_\ell}.$$

- Difference set *naturally* emerges from minimum distance of sensing subspace codes

$$\mathcal{A}-\mathcal{A}\stackrel{\text{def}}{=}\{a-a':a,a'\in\mathcal{A},a\neq a'\}.$$

- Difference sets also known to arise in other context, including...
  - Frames w. harmonic structure (Xia, Zhou, and Giannakis (2005); Fickus, Iverson, Jasper, and King (2021))
  - *Sparse array* design and array signal processing (correlation priors) (Pal and Vaidyanathan, 2015)

Large minimum distance using (modulo) Golomb rulers

How to find sensing subspace codes $\mathcal{C}$ with large $d_{\min}^{(s)}(\mathcal{C})$ when $N \gg M$?

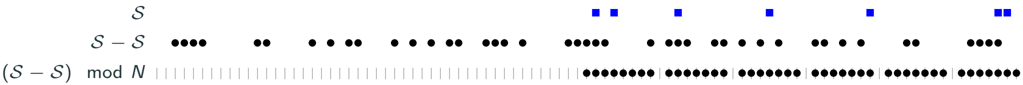
Leverage insights from difference set (sparse array) design!

- For instance: If $(\mathcal{A} - \mathcal{A}) \mod N$ covers *almost all* elements in $[N]$ *exactly once*, then

$$\max_{\substack{\alpha \in \mathcal{Z}(N) \\ \alpha \neq 1}} \sum_{d \in \mathcal{A} - \mathcal{A}} \alpha^d \approx 0 \implies d_{\min}^{(s)}(\mathcal{C}) \approx 1, \quad \text{since} \quad \sum_{d \in [N]} \alpha^d = 0, \ \forall \alpha \in \mathcal{Z}(N) \setminus \{1\}$$

- Finding such an $\mathcal{A} = \{d_1, d_2, \ldots, d_M\}$ is a (modulo) Golomb ruler design problem
- We consider the Bose-Chowla construction (Bose and Chowla, 1962; Dimitromanolakis, 2002)

Bose-Chowla (Golomb) ruler ($M = 7$, $N = 48$)



Lower bound on minimum distance of Bose-Chowla code

Theorem (Minimum distance of Bose-Chowla code)

For Bose-Chowla sensing subspace code $\mathcal{C}^{(BC)}$, with M prime and $N = M^2 - 1$, we have

$$d_{\text{min}}^{(s)}(\mathcal{C}^{(BC)}) > 1 - \frac{2}{M}.$$

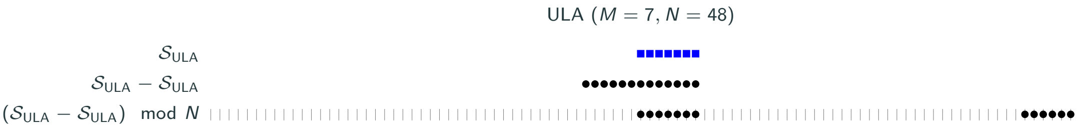
$d_{\text{min}}^{(s)}(\mathcal{C}^{(\text{BC})})$ approaches *optimal value* of 1 at practically *optimal rate* (Welch bound $\approx 1 - \frac{1}{\sqrt{N}}$)

Suboptimality of uniform sampling

- Uniform linear array (ULA) dominant array geometry in sensing and comms for decades

$$\mathcal{S}_{\mathsf{ULA}} \stackrel{\text{def}}{=} [M] - 1. \tag{6}$$

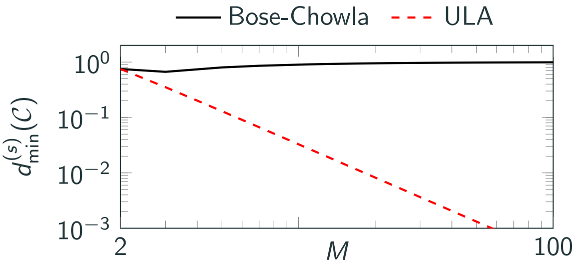
- Simply Nyquist sampling: well-understood, amenable to analysis & efficient algorithms



Using the sensing subspace code defined by the ULA is a bad idea

Can show that $d_{\min}^{(s)}(\mathcal{C}^{(\mathsf{ULA})}) \leqslant 1 - \frac{4}{\pi^2} < 0.6$ when $N \approx M^2$; suboptimal!

Minimum distance of sensing subspace codes



As code length M (# of sensors) increases, min. distance $\rightarrow \begin{cases} 1, \text{in case of BC} \\ 0, \text{in case of ULA} \end{cases}$

Why care about minimum distance?

The maximum-likelihood decoder reduces to the minimum distance decoder, $\mathcal{D}_{\mathrm{min}}$:

$$\hat{\alpha} = \underset{\alpha:\alpha\in\mathcal{Z}(N)}{\arg\max} |Y^{\mathrm{H}} \boldsymbol{c}(\alpha)|, \tag{7}$$

where $\mathcal{Z}(N) \overset{\mathrm{def}}{=} \{e^{j2\pi n/N}, n\in[N]-1\}$ denotes the set of N-th roots of unity

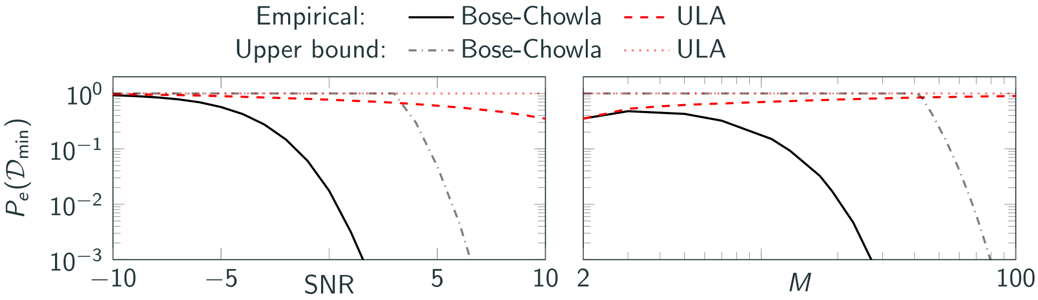
Lemma (Probability of error of $\mathcal{D}_{\mathrm{min}}$)

For a one-dimensional sensing subspace code $\mathcal{C}$ with minimum subspace distance $d_{\mathrm{min}}^{(s)}(\mathcal{C})$,

$$P_e(\mathcal{D}_{\mathrm{min}}) \overset{\mathrm{def}}{=} \Pr\{\hat{\theta} \neq \theta\} \leqslant \exp\left(-\frac{M}{4\sigma^2}\left(1-\sqrt{1-d_{\mathrm{min}}^{(s)}(\mathcal{C})}\right)^2 + \ln N\right). \tag{8}$$

Large minimum distance ensures low probability of decoding error

Error probability of maximum-likelihood decoder



- Bose-Chowla ruler achieves a low P_e both with increasing SNR (left) or code length (right)
- ULA is significantly less robust to noise due to the smaller minimum codeword distance

Contributions: Bridging subspace coding and sensing

We establish first contact between the two fields of subspace coding and sensing, showing that they share many fundamental objectives despite having evolved independently

Main contributions

1. Classical DoA estimation using multisensor array maps into a subspace coding problem
   - Resulting *"sensing subspace codes"* (in $\mathbb{C}^N$) have particular *structure imposed by nature*
2. Sensing and subspace coding have core mathematical problems in common
   - Maximizing min. distance of sensing subspace code $\leftrightarrow$ designing noise-robust sensor arrays
3. Insights & methodologies in one field can be leveraged to tackle problems in the other
   - Difference sets inform design of both (sparse) sensor arrays and (sensing) subspace codes

https://arxiv.org/pdf/2407.02963

Join us at Tampere University (TUNI)!

Interested in doctoral studies at TUNI (Signal Processing and Machine Learning)?

Come talk to me or reach out per email: *robin.rajamaki@tuni.fi*

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